Butson-type complex Hadamard matrices and association schemes on the Galois rings of characteristic 4

Takuya Ikuta (Kobe Gakuin University, Japan)
(joint work with Akihiro Munemasa (Tohoku University, Japan))

5th workshop on real and complex Hadamard matrices and application
10 July, 2017, Budapest

1 motivation

$$\mathfrak{X} = (X, \{R_j\}_{i=0}^{d})$$

commutative association scheme

one to one correspondence ⟷

$$\mathfrak{A} = \langle A_i \mid i = 0, \ldots, d \rangle$$

Bose-Mesner algebra

adjacency matrices

$$\sum_{i=0}^{d} A_i = J,$$

$$A_i A_j = A_j A_i,$$

$$A_i^T = A_j \quad (j \in \{0, 1, \ldots, d\}),$$

$$A_i A_j \in \mathfrak{A}.$$

Problem:

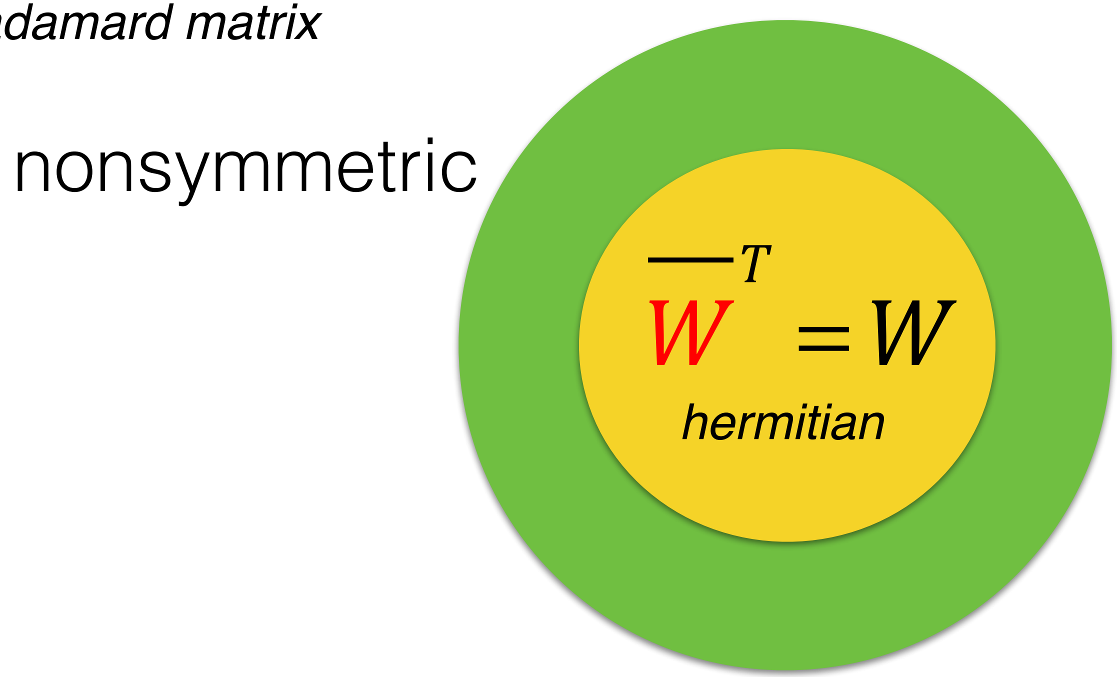
Let $W = A_0 + w_1 A_1 + \cdots + w_d A_d \in \mathfrak{A},$

$$|w_1| = \cdots = |w_d| = 1.$$

Find W *such that* $W \overline{W}^T = |X| I.$

	Examples (symmetric)	Examples (nonsymmetric)
$d=3$	*Complex Hadamard matrices contained in a Bose-Mesner algebra,* Spec.Matrices.,3 (2015), 91-110.	?
$d=4$	*Complex Hadamard matrices attached to even orthogonal scheme of class 4,* (2016), submitted.	

not Butson-type complex Hadamard matrix



2 6-class nonsymmetric association scheme on the Galois ring of characteristic 4

$e \geq 3$: an *odd* positive integer

$$\mathrm{GF}(2^e) = \frac{\mathrm{GF}(2)[x]}{(\varphi(x))},$$

$\varphi(x)$: a primitive polynomial of degree e over $\mathrm{GF}(2)$

$$\mathrm{GF}(2^e)^\times = \langle \zeta \rangle$$

$$\mathbb{Z}_4 = \mathbb{Z} / 4\mathbb{Z}$$

$\exists \Phi(x)$: a monic polynomial of degree e over $\mathbb{Z}_4$ s.t.

$$\begin{cases} \Phi(x) \equiv \varphi(x) \mod 2\mathbb{Z}_4[x], \\ \Phi(x) \mid x^{2^e-1} - 1 \text{ in } \mathbb{Z}_4[x]. \end{cases}$$

$$\mathfrak{R} = \frac{\mathbb{Z}_4[x]}{(\Phi(x))}$$: *Galois ring*

$$\wp = 2\mathfrak{R}.$$

radical

$$|\mathfrak{R}| = 4^e \qquad |\wp| = 2^e$$

ξ: the image of x in $\mathfrak{R} = \dfrac{\mathbb{Z}_4[x]}{(\Phi(x))}$.

$\mathfrak{R} = \mathbb{Z}_4[\xi]$,

$\mathfrak{T} = \langle \xi \rangle \qquad |\mathfrak{T}| = 2^e - 1.$

$\mathfrak{R} =$ $\mathfrak{R}^*$ the unit group $= \mathfrak{T}\mathfrak{E}$; the principal unit group $\mathfrak{E} = 1 + \wp$; $\wp$ radical

$\mathfrak{T} = \langle \xi \rangle$,

$|\mathfrak{E} : 1 + \wp_0| = 2$

$1 + \wp_0$

$|\wp : \wp_0| = 2$

$\mathfrak{T}_0 = \{ \xi^j \mid 0 \le j \le 2^e - 2,\ \mathrm{Tr}(\zeta^j) = 0 \}$

$\zeta^j \in \mathrm{GF}(2^e)$

$\mathfrak{T}_0$

$\mathfrak{E} = (1 + \wp_0) \cup (-(1 + \wp_0))$

$\wp_0 = 2\mathfrak{T}_0 \cup \{0\}$.

$(\mathfrak{T} \setminus \{1\})(1 + \wp_0)$,	$1 + \wp_0$	$\{0\}$	$\wp_0 \setminus \{0\}$
$(\mathfrak{T} \setminus \{1\})(-(1 + \wp_0))$,	$-(1 + \wp_0)$		$\wp \setminus \wp_0$

$\mathfrak{R}^*$ $\wp$

$S_1 = (\mathfrak{T} \setminus \{1\})(1+\wp_0)$, $S_3 = 1+\wp_0$,

$S_2 = (\mathfrak{T} \setminus \{1\})(-(1+\wp_0))$, $S_4 = -(1+\wp_0)$.

$S_0 = \{0\}$, $S_5 = \wp_0 \setminus \{0\}$,

$S_6 = \wp \setminus \wp_0$.

$$\mathfrak{R} = S_0 \cup S_1 \cup \cdots \cup S_6, \ S_i \cap S_j = \emptyset.$$

$$\lambda_\alpha(A) = \sum_{\beta \in A} \chi(\alpha\beta) \quad (\alpha \in \mathfrak{R}, \ A \subset \mathfrak{R})$$

$$\chi(\alpha) = i^{eS(\alpha)}$$

trace from $\mathfrak{R}$ to $\mathbb{Z}_4$

Theorem1 (Munemasa-I)

$\lambda_\alpha(S_j)$ is constant for $\forall \alpha \in S_i$ $(i = 0,1,\ldots,6)$.

Put $p_{i,j} = \lambda_\alpha(S_j)$ for $\alpha \in S_i$ $(i,j = 0,1,\ldots,6)$. Then

$$(p_{i,j})_{0\le i,j\le 6} = \begin{array}{c} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{array} \begin{array}{ccccccc} S_0 & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \\ \end{array}$$

	S_0	S_1	S_2	S_3	S_4	S_5	S_6
S_0	1	$2b(b-1)$	$2b(b-1)$	b	b	$b-1$	b
S_1	1	bi	$-bi$	0	0	-1	0
S_2	1	$-bi$	bi	0	0	-1	0
S_3	1	0	0	bi	$-bi$	$b-1$	$-b$
S_4	1	0	0	$-bi$	bi	$b-1$	$-b$
S_5	1	$-2b$	$-2b$	b	b	$b-1$	b
S_6	1	0	0	$-b$	$-b$	$b-1$	b

$$b = 2^{e-1}, \ i^2 = -1.$$

$\mathfrak{R}^*$ $\wp$

Schur ring

$$S_1 = (\mathfrak{T} \setminus \{1\})(1+\wp_0), \quad S_3 = 1+\wp_0, \quad S_0 = \{0\}, \; S_5 = \wp_0 \setminus \{0\},$$
$$S_2 = (\mathfrak{T} \setminus \{1\})(-(1+\wp_0)), \quad S_4 = -(1+\wp_0). \quad S_6 = \wp \setminus \wp_0.$$

$$R_j = \left\{ \left(\alpha, \beta \right) \in \mathfrak{R} \times \mathfrak{R} \;\mid\; \alpha - \beta \in S_j \right\} \quad \left(j = 0,1,\ldots,6 \right)$$

$\mathfrak{X} = (\mathfrak{R}, \{R_j\}_{j=0}^{6})$: 6-class nonsymmetric association scheme

$$A_j \leftrightarrow R_j$$

adjacency matrix

$$A_1^T = A_2, \; A_3^T = A_4, \; A_5, \; A_6$$

symmetric

3 hermitian complex Hadamard matrices attached to a 6-class nonsymmetric association scheme on the Galois ring of characteristic 4

$$A_1^T = A_2,\ A_3^T = A_4,\ A_5,\ A_6$$

symmetric

$\mathfrak{A} = \langle A_0, A_1, \ldots, A_6 \rangle$: Bose—Mesner algebra

Assume that

- $W = A_0 + w_1 A_1 + \cdots + w_6 A_6 \in \mathfrak{A}$, $|w_1| = \cdots = |w_d| = 1$.
- W: *hermitian*

Theorem2 (Munemasa-Ikuta)

Let $W = A_0 + w_1 A_1 + \cdots + w_6 A_6 \in \mathfrak{A}$,

hermitian $|w_1| = \cdots = |w_d| = 1$.

W: *a complex Hadamard matrix*

if and only if $W = A_0 + \varepsilon_1 i(A_1 - A_2) + \varepsilon_2 i(A_3 - A_4) + A_5 + A_6$, *or,*

$W = A_0 + \varepsilon_1 i(A_1 - A_2) + \varepsilon_2 (A_3 + A_4) + A_5 - A_6$,

for some $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$.

How to prove?

$$\mathfrak{A} = \langle A_0, A_1, \ldots, A_d \rangle = \langle E_0, E_1, \ldots, E_d \rangle$$

adjacency matrices *primitive idempotents*

$$W = \sum_{j=0}^{d} w_j A_j = \sum_{j=0}^{d} w_j \left(\sum_{i=0}^{d} P_{i,j} E_j \right)$$

$$= \sum_{i=0}^{d} \left(\sum_{j=0}^{d} w_j P_{i,j} \right) E_i$$

W : hermitian

$$\Rightarrow \quad W\overline{W}^T = W^2 = \sum_{i=0}^{d} \left(\sum_{j=0}^{d} w_j P_{i,j} \right)^2 E_i$$

$$\|$$

$$nI = n(E_0 + E_1 + \cdots + E_d)$$

$$\left(\sum_{j=0}^{d} w_j P_{i,j} \right)^2 = n$$

Thank you very much for listening to my talk.