Gröbner bases and cocyclic Hadamard matrices

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5th Workshop on Real and Complex Hadamard Matrices and Applications

Outline

Hadamard cocyclic ideals

Hadamard ideals - Kotsireas, Koukouvinos, Seberry (2006)

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G is a group of order $4t$, a *cocycle* ψ over G is a mapping $\psi : G \times G \to \langle -1 \rangle$ satisfying $\psi(1,1) = \psi(g,1) = \psi(1,g) = 1,\ g \in G$ and the cocycle equation:

$$\psi(g_i, g_j)\ \psi(g_i g_j, g_k) \psi(g_i, g_j g_k)\ \psi(g_j, g_k) = 1, \quad g_i, g_j, g_k \in G. \qquad (1)$$

Pros & Cons

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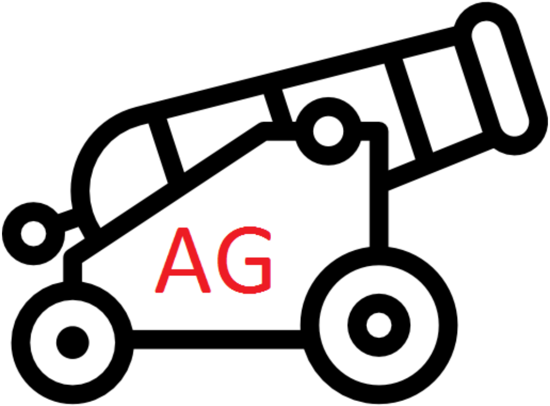
Cons

- $\{$ Cocyclic Hadamard Matrices $\} \subset \{$ Hadamard Matrices $\}$

The idea

Use the Algebraic Geometry artillery (namely Gröbner basis techniques) to determine both the cardinality and the elements of the set $\mathcal{H}_G$ of cocyclic Hadamard matrices over a multiplicative finite group G of $4t$ elements.



First (naive) approach

$\mathbb{Q}[X_G]$ be the polynomial ring over $\{X_G\} = \{x_{i,j} \colon\ g_i, g_j \in G\}$

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Theorem

The set $\mathcal{H}_G$ can be identified with the set of zeros of the zero-dimensional ideal $I_G = I_G^1 + I_G^2 + I_G^3 + I_G^4 \subset \mathbb{Q}[X_G]$, where

$$\begin{cases} I_G^1 = \langle\, x_{i,j}^2 - 1 \colon \ i,j \in G \,\rangle \\ I_G^2 = \langle\, x_{i,j}x_{ij,k} - x_{j,k}x_{i,jk} \colon \ i,j,k \in G \,\rangle \\ I_G^3 = \langle\, x_{1,j} - 1, x_{i,1} - 1 \colon \ i,j \in G \setminus \{1\} \,\rangle \\ I_G^4 = \langle\, \sum_{j \in G} x_{i,j} \colon \ i \in G \setminus \{1\} \,\rangle \end{cases}$$

Complexity

Set of polynomials defining I_G:

$$O(t^3) \text{ polynomials of degree up to 2 over } O(t^2) \text{ variables}$$

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Open computer algebra system for polynomial computations SINGULAR
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$$\begin{cases} G = 1 \Rightarrow \mathbb{Z}_t \times \mathbb{Z}_2^2, G = 2 \Rightarrow D_{4t} \\ opt = 1 \Rightarrow \sharp\mathcal{H}_G, opt = 2 \Rightarrow \mathcal{H}_G \end{cases}$$

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$$t \leq 3$$

Basis of normalized cocycles

Fixed a representative cocycle ρ, and a basis for normalized cocycles **B**.
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*(Álvarez et al.)[2008] The matrix M_ψ is Hadamard if and only if the vector of coordinates $(x_1, \ldots, x_k)_{\mathbf{B}}$ of ψ with regards to **B** satisfies the following system of $4t-1$ equations and k unknowns*

$$\left\{ \begin{array}{lcl} (m_{2,1}^1)^{x_1} \ldots (m_{2,1}^k)^{x_k} + \ldots + (m_{2,4t}^1)^{x_1} \ldots (m_{2,4t}^k)^{x_k} & = & 0 \\ & \vdots & \\ (m_{4t,1}^1)^{x_1} \ldots (m_{4t,1}^k)^{x_k} + \ldots + (m_{4t,4t}^1)^{x_1} \cdots (m_{4t,4t}^k)^{x_k} & = & 0 \end{array} \right. \tag{2}$$

□

Theorem

The set $\mathcal{H}_G^\rho$ can be identified with the set of zeros of the following zero-dimensional ideal of $\mathbb{Q}[X]$.

$$J_G := \langle\, x_i^2 - x_i : \ i \in \{1, \ldots, k-m\}\,\rangle + \langle \sum_{h=1}^{4t} s_{l,h}(X) : \ l \in \{1, \ldots 4t-1\}\,\rangle.$$

$s(l,h)$ is defined in terms of paths and intersections, and $\deg(s_{l,h}) \leq 2$.

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$$t \leq 7, \; D_{4t}$$

Fine tuning: $\mathbb{Z}_t \times \mathbb{Z}_2^2$

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Diagrammatic properties:

$$\left|\begin{array}{ccccccc} - & - & - & \times & \times & \times & - \\ - & - & \times & - & \times & - & \times \\ - & - & - & - & \times & - & - \\ \times & \times & - & - & \times & - & - \end{array}\right|$$

$\texttt{CocAH}(t, col, dist, H)$

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$\texttt{CocAH}(t, col, dist, H)$

$$\begin{cases} \text{col: parity of columns} \\ \text{dist: sum of each row} \\ \text{H: fixed values of some coordinates} \end{cases}$$

Fine tuning: $\mathbb{Z}_t \times \mathbb{Z}_2^2$, example

$t = 31$

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col: $0, 4, 2, 4, 2, 2, 2, 2, 0, 2, 2, 0, 4, 2, 2$

Fine tuning: $\mathbb{Z}_t \times \mathbb{Z}_2^2$, example

$t = 31$

col: $0, 4, 2, 4, 2, 2, 2, 2, 0, 2, 2, 0, 4, 2, 2$

dist: $12, 18, 18, 12$

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$$10F6F96990660F6666F06609969F6F0$$

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$\texttt{CocDH}(t, col, dist, H)$

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$$\begin{cases} \text{dist: number of new intersections in each row [2,t]} \\ \text{H: fixed values of some coordinates} \end{cases}$$

Fine tuning: D_{4t}, example

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dist $1, 1, 2, 2, 3, 3, 2, 3, 0, 1, 0, 4, 2, 2, 3, 2, 3, 2, 2, 0, 0, 3, 1, 1, 2, 1, 3, 2, 2, 3$

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$H = \underline{5}, \underline{6}, \underline{17}, \underline{36}, \underline{48}, 63, 64, \underline{84}, \underline{95}, 115, \underline{117}$

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$$21172424E984E2E3FC6B06D5527CA70$$

Future work

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Improve the quality of the helping information

to get better results !

Thank you and farewell !!!