Any SIC-POVM defines a complex Hadamard matrix in every dimension

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POVM

Positive Operator Valued Measure

Most general measurements in quantum mechanics

A set of N rank-one operators $\{\Pi_k\}$ form a POVM if

1. Π_k is positive semi-definite, $k = 0, \dots N-1$

2. $\sum_{k=0}^{N-1} \Pi_k = \mathbb{1}$

E.g.: Orthonormal bases $\{|\phi_k\rangle\}$, $\Pi_k = |\phi_k\rangle\langle\phi_k|$

(Tight frames in Mathematics)

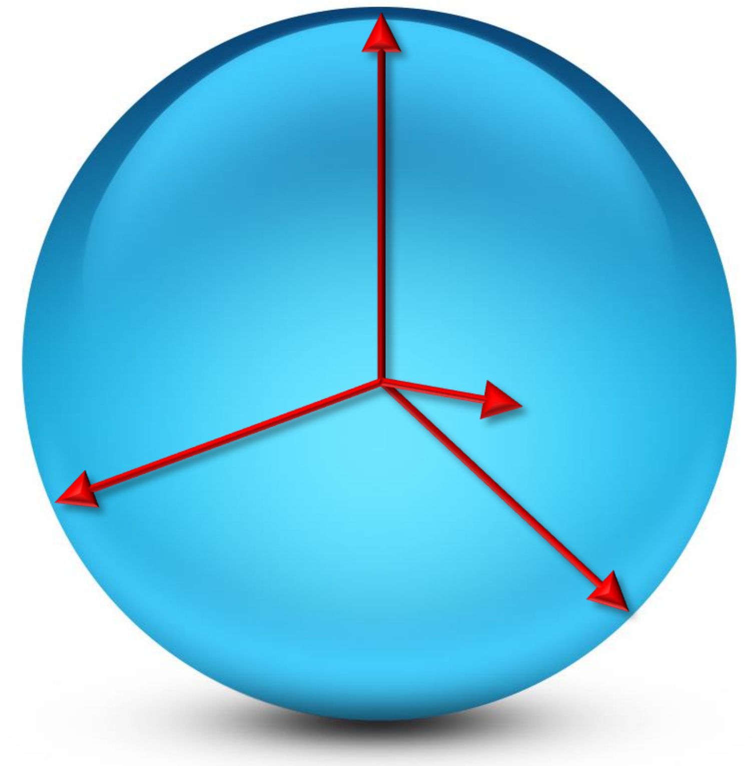
Equiangular tight frames

N vectors $\{\phi_j\}$ in dimension d such that

$$\left|\langle\phi_j|\phi_k\rangle\right|^2 = \frac{N-d}{d(N-1)} \quad (j \neq k)$$

ETF for $N = d^2$ are known as SIC-POVM

Bloch sphere



$$\phi_0 = \binom{1}{0}, \phi_1 = \frac{1}{\sqrt{3}}\binom{1}{\sqrt{2}}; \;\; \phi_2 = \frac{1}{\sqrt{3}}\binom{1}{\omega\sqrt{2}}, \phi_3 = \frac{1}{\sqrt{3}}\binom{1}{\omega^2\sqrt{2}}, \qquad \omega = e^{2\pi i/3}$$

Covariant SIC-POVM

Conjecture :

In every dimension d there exists a fiducial rank-one projector Π_0 such that the set $\{D_p \Pi_0 D_p^\dagger\}_{p \in Z_d^2}$ defines a SIC-POVM.

$$D_p = \tau^{p_1 p_2} X^{p_1} Z^{p_2}$$

$$X|k\rangle = |[k+1]\rangle$$

$$Z|k\rangle = \omega^k |k\rangle$$

$$p = (p_1, p_2) \in \mathbb{Z}_d^2$$

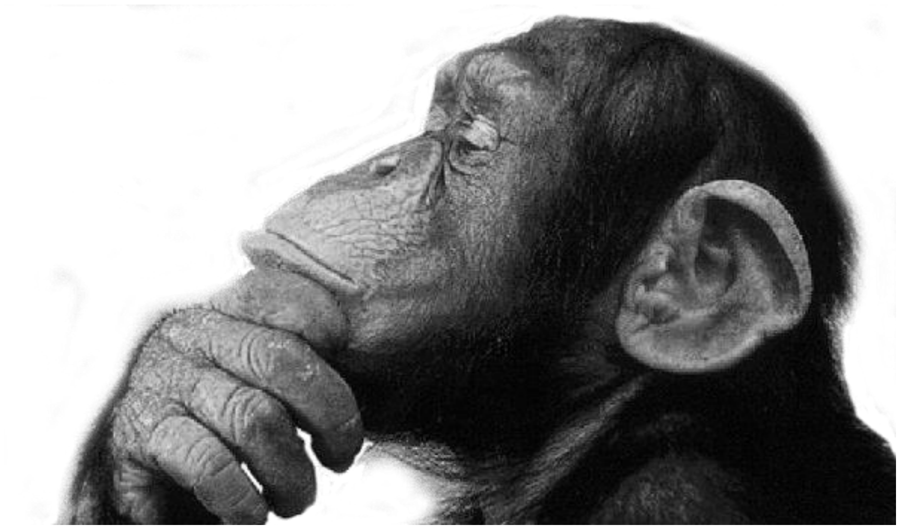
$$\tau = -e^{\pi i/d} \qquad \omega = e^{2\pi i/d}$$

G. Zauner, Quantendesigns: Grundzüge einer nichtkommutativen Designtheorie, Ph.D. Thesis, University of Vienna, 1999

Existence of SIC-POVM

Analytical: $d = 2 - 21, 24, 35, 48$

Numerical: $d = 2 - 121$



Open in prime dimensions!

Every known SIC-POVM in dimension d is covariant under $WH(d)$, except the so-called *Hoggar lines in* $d = 8$, which are covariant under $WH(2)^{\otimes 3}$.

SIC-POVM and complex Hadamard matrices

SIC-POVM and Complex Hadamard matrices (I)

Let

$$\Pi_l = \lambda\Lambda_0 + \nu \sum_{k=0}^{d^2-1} u_{kl}\Lambda_k$$

be the linear decomposition of a set of d^2 rank-one projectors Π_l on the orthonormal basis $\{\Lambda_k\}$. The set $\{\Pi_k\}$ forms a SIC-POVM if and only if the matrix U having entries u_{kl} is unitary.

$$\Lambda_0 = \mathbb{I}$$

$$\mathrm{Tr}(\Lambda_i\Lambda_j^\dagger) = c\,\delta_{i,j}$$

$$\nu = \frac{d}{c\sqrt{d+1}}$$

$$\lambda = \frac{1}{d} + \frac{1}{d\sqrt{d+1}}$$

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Good news

Covariant SIC-POVM in dimension d has associated a CHM $U \sim F_d \times F_d$.

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Bad news

To find U from $F_d \times F_d$ is equivalent to find the fiducial state.

Finding covariant SIC-POVM from CHM (I)

1 isolated SIC-POVM in dimension $d = 2$: $F_2 \times F_2 \;\rightarrow\; U$

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16 isolated SIC-POVM in dimension $d = 4$: $F_4 \times F_4 \;\rightarrow\; U_1, \dots, U_{16}$

Complex Hadamard matrices and SIC-POVM (II)

G

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(Analytic proof in every dimension where a SIC-POVM exists)

It comes from the fact that a SIC-POVM is a SIC-POVM, and nothing more than that.

Complex Hadamard matrices and SIC-POVM (II)

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Roughly speaking

$$SIC - POVM = \sqrt{[CHM]}$$

“CHM are the roots of the SIC-POVM problem”

SIC-POVM in *d=2*

$$H = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

$$\sqrt{H} = \begin{pmatrix} 1 & -i(-1)^{a_1} & -i(-1)^{a_2} & -i(-1)^{a_3} \\ i(-1)^{a_1} & 1 & -i(-1)^{a_4} & -i(-1)^{a_5} \\ i(-1)^{a_2} & i(-1)^{a_4} & 1 & -i(-1)^{a_6} \\ i(-1)^{a_3} & i(-1)^{a_5} & i(-1)^{a_6} & 1 \end{pmatrix} \qquad a_k = 0{,}1$$

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$a_k = 0,1$

a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	1	0	1
0	1	0	0	0	0
0	0	0	0	1	0
0	1	0	1	1	1
0	0	1	0	0	1
0	1	1	1	0	0
0	0	1	1	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	0	0	1
1	0	0	0	1	1
1	1	0	1	1	0
1	0	1	0	0	0
1	1	1	1	0	0
1	0	1	1	1	1
1	1	1	0	1	0

16 SOLUTIONS

SIC-POVM in *d=3*

Proposition

Let H be an hermitian complex Hadamard matrix of size 9 having constant diagonal. Therefore, $H' = H \circ H$ is a complex Hadamard matrix.

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Result 1	There is a 1-parametric family of matrices H of size 9 (subfamily of the Fourier family $F_9^{(4)}$).
Result 2	There is no isolated matrix H.

SIC-POVM in *d=3*

$$H(\alpha) =$$

$$\begin{pmatrix}
1 & 1+\omega^2 & 1-\omega & e^{-I\alpha} & -e^{-I\alpha}\omega & e^{-I\alpha}\omega^2 & e^{I\alpha} & -e^{I\alpha}\omega & e^{I\alpha}\omega^2 \\
1-\omega & 1 & 1+\omega^2 & -e^{-I\alpha}\omega & e^{-I\alpha}\omega^2 & e^{-I\alpha} & e^{I\alpha} & -e^{I\alpha}\omega & e^{I\alpha}\omega^2 \\
1+\omega^2 & 1-\omega & 1 & e^{-I\alpha}\omega^2 & e^{-I\alpha} & e^{-I\alpha}\omega & e^{I\alpha} & -e^{I\alpha}\omega & e^{I\alpha}\omega^2 \\
e^{I\alpha} & e^{I\alpha}\omega^2 & -e^{I\alpha}\omega & 1 & 1-\omega & 1+\omega^2 & e^{-I\alpha} & e^{-I\alpha}\omega^2 & -e^{-I\alpha}\omega \\
e^{I\alpha}\omega^2 & -e^{I\alpha}\omega & e^{I\alpha} & 1+\omega^2 & 1 & 1-\omega & e^{-I\alpha} & e^{-I\alpha}\omega^2 & -e^{-I\alpha}\omega \\
-e^{I\alpha}\omega & e^{I\alpha} & e^{I\alpha}\omega^2 & 1-\omega & 1+\omega^2 & 1 & e^{-I\alpha} & e^{-I\alpha}\omega^2 & -e^{-I\alpha}\omega \\
e^{-I\alpha} & e^{-I\alpha} & e^{-I\alpha} & e^{I\alpha} & e^{I\alpha} & e^{I\alpha} & 1 & -1 & -1 \\
e^{-I\alpha}\omega^2 & e^{-I\alpha}\omega^2 & e^{-I\alpha}\omega^2 & -e^{I\alpha}\omega & -e^{I\alpha}\omega & -e^{I\alpha}\omega & -1 & 1 & -1 \\
-e^{-I\alpha}\omega & -e^{-I\alpha}\omega & -e^{-I\alpha}\omega & e^{I\alpha}\omega^2 & e^{I\alpha}\omega^2 & e^{I\alpha}\omega^2 & -1 & -1 & 1
\end{pmatrix}$$

$$\omega = e^{2\pi i/3}$$

SIC-POVM in *d=3*

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\end{pmatrix}$$

$$\omega = e^{2\pi i/3}$$

SIC-POVM in *d=4*

One parametric family of CHM for any unimodular complex number *a*.

SIC-POVM only for

$$a = \sqrt{\frac{3}{2} - \frac{\sqrt{5}}{2}} + \sqrt{-\frac{1}{2} + \frac{\sqrt{5}}{2}}\, i$$

SIC-POVM $\{D_p \phi_{4a}\}$

Scott and Grassl,
J. Math. Phys. 51, 042203 (2010)

$$\begin{pmatrix}
1 & -a^* & 1 & -a & -a^* & -a & -a & -a & 1 & -a^* & 1 & -a & -a & -a^* & -a^* & -a^* \\
-a & 1 & -a^* & 1 & ia & ia^* & ia & ia & a & -1 & a^* & -1 & -ia^* & -ia & -ia^* & -ia^* \\
1 & -a & 1 & -a^* & a & a & a^* & a & 1 & -a & 1 & -a^* & a^* & a^* & a & a^* \\
-a^* & 1 & -a & 1 & -ia & -ia & -ia & -ia^* & a^* & -1 & a & -1 & ia^* & ia^* & ia^* & ia \\
-a & -ia^* & a^* & ia^* & 1 & -ia^* & -1 & ia & -a^* & -ia & a & ia & 1 & -ia^* & -1 & ia \\
-a^* & -ia & a^* & ia^* & ia & 1 & -ia^* & -1 & a & ia^* & -a & -ia & -ia & -1 & ia^* & 1 \\
-a^* & -ia^* & a & ia^* & -1 & ia & 1 & -ia^* & -a & -ia & a^* & ia & -1 & ia & 1 & -ia^* \\
-a^* & -ia^* & a^* & ia & -ia^* & -1 & ia & 1 & a & ia & -a & -ia^* & ia^* & 1 & -ia & -1 \\
1 & a^* & 1 & a & -a & a^* & -a^* & a^* & 1 & a^* & 1 & a & -a^* & a & -a & a \\
-a & -1 & -a^* & -1 & ia^* & -ia & ia^* & -ia^* & a & 1 & a^* & 1 & -ia & ia^* & -ia & ia \\
1 & a & 1 & a^* & a^* & -a^* & a & -a^* & 1 & a & 1 & a^* & a & -a & a^* & -a \\
-a^* & -1 & -a & -1 & -ia^* & ia^* & -ia^* & ia & a^* & 1 & a & 1 & ia & -ia & ia & -ia^* \\
-a^* & ia & a & -ia & 1 & ia^* & -1 & -ia & -a & ia^* & a^* & -ia^* & 1 & ia^* & -1 & -ia \\
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\end{pmatrix}$$

SIC-POVM from CHM (II)

To find a SIC-POVM from a given H we have to find a suitable combination out of $2^{d^2(d^2-1)/2}$ possible ways

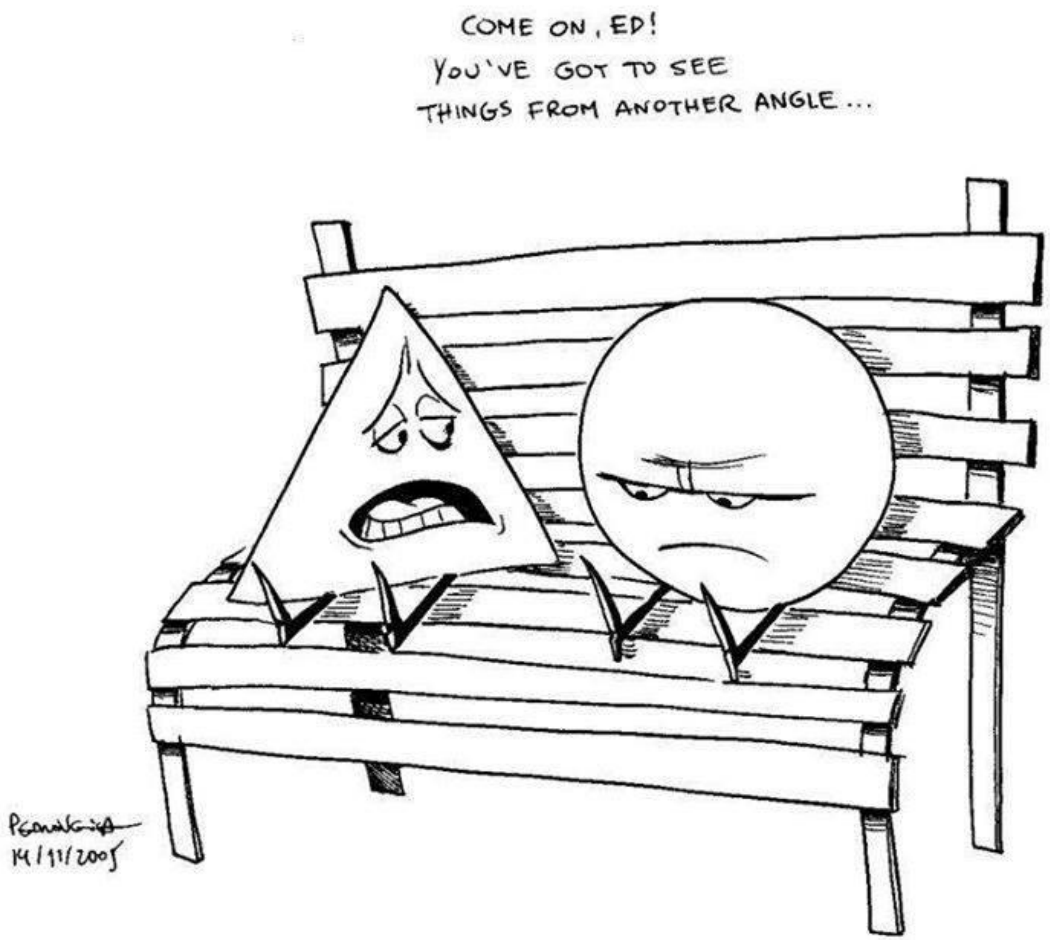
Dimension (*d*)	**# Combinations**
2	2^6
3	2^{36}
4	2^{120}

Conclusion

Complex Hadamard matrices might be the key to unlock the SIC-POVM problem

So many keys

So many padlocks



Thank you for your attention