Quasi-orthogonal cocyclic matrices

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(with José Andrés Armario)

5th Workshop on Real and Complex Hadamard Matrices and Applications

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For $\mathrm{BH}(n, k)$, distinct rows (resp., columns) are pairwise orthogonal.

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Combining with orthogonality adds further constraints on parameters of the design; e.g., if $\mathrm{H}(n)$ is group-developed then n must be a square.

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G, U groups, U abelian. $Z^2(G,U) :=$ all maps $\psi : G \times G \to U$ such that

$$\psi(x,y)\psi(xy,z) = \psi(x,yz)\psi(y,z) \quad \forall\, x,y,z \in G;$$

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ψ is orthogonal $\Leftrightarrow$ no. $+1$s $=$ no. -1s in every non-initial row of M_ψ.

That is, the *row excess* $\mathrm{RE}(M) := \sum_{i\geq 2} |\sum_{j\geq 1} m_{i,j}|$ of a cocyclic Hadamard matrix $M = [m_{i,j}]$ is optimal (least, i.e., zero).

Analogs of orthogonal cocycle for orders $\not\equiv 0$ modulo 4

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Lemma

Let M be a cocyclic $\{\pm 1\}$-matrix with indexing group G.

(i) *no. even rows of M is $4t + 2$ or $2t + 1$; so $\mathrm{RE}(M) \geq 4t$.*

(ii) $\mathrm{RE}(M) = 4t$ *if and only if*

$$\mathrm{abs}(MM^\top) = \begin{bmatrix} 4tI + 2J & 0 \\ 0 & 4tI + 2J \end{bmatrix}$$

up to row permutation ($\mathrm{abs}([x_{ij}]) = [|x_{ij}|]$, J is all 1s matrix).

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Lemma

ψ is quasi-orthogonal $\iff$ $|\{g \in G \setminus \{1\} \mid \sum_{h \in G} \psi(g,h) = \pm 2\}| = 2t$ *and* $|\{g \in G \setminus \{1\} \mid \sum_{h \in G} \psi(g,h) = 0\}| = 2t + 1$.

Ehlich-Wojtas bound for $(4t+2) \times (4t+2)$ $\{\pm 1\}$-matrices M: $|\det(M)| \leq 2(4t+1)(4t)^{2t}$. For the bound to be attained, $4t+1$ must be the sum of two squares.

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If M_ψ for $\psi \in Z^2(G, \langle -1 \rangle)$ attains the Ehlich-Wojtas bound then ψ is quasi-orthogonal.

Example. $\psi \in Z^2(\mathrm{Sym}(3), \langle -1 \rangle)$ given by

$$M_\psi = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 \end{bmatrix}$$

is quasi-orthogonal. $\det(M_\psi) = 128$ does not attain the E-W bound.

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If $|G| = 4t + 2$ *and* $\psi \in Z^2(G, \langle -1 \rangle)$ *is a coboundary then* ψ *is not quasi-orthogonal*

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Also, haven't yet found G of an allowable order for which there are no quasi-orthogonal cocycles whose matrices attain the E-W bound.

For $\psi \in Z^2(G,U)$, E_ψ denotes the central extension $\{(g,u) \mid g \in G, u \in U\}$ of U by G with multiplication $(g,u)(h,v) = (gh, uv\,\psi(g,h))$.

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$\mathcal{E}_{M_\psi}$ (really $\frac{1}{2}(\mathcal{E}_{M_\psi} + J_{8t})$) is an incidence matrix of a special block design.

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As well as the points and blocks being permuted regularly by E_ψ (i.e., $\mathcal{E}_{M_\psi}$ is group-developed over E_ψ), the central subgroup $\langle(1,-1)\rangle$ of E_ψ acts regularly on each point class.

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In summary

Theorem

If ψ is orthogonal then $\mathcal{E}_{M_\psi}$ is a divisible $(4t, 2, 4t, 2t)$-design with E_ψ as a regular group of automorphisms, which is class regular wrt $\langle(1,-1)\rangle \subseteq E_\psi$.

Let E be a group of order vu with a subgroup U of order u. A k-set $R \subseteq E$ is a (v, u, k, λ)-*relative difference set* (wrt the *forbidden subgroup* U) if the multiset $\{r_1 r_2^{-1} \mid r_1, r_2 \in R,\, r_1 \neq r_2\}$ contains each element of $E \setminus U$ precisely λ times, and contains no element of U.

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So: orthogonal cocycle $\equiv$ Hadamard group $\equiv$ central RDS $\equiv$ class regular divisible design.

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The analog of [orthogonal cocycle $\equiv$ class regular divisible design (with regular Hadamard group of automorphisms)] is more interesting.

Let X be a v-set and $R_0 = \{(x, x) \mid x \in X\}, R_1, \ldots, R_m$ nonempty subsets of $X \times X$, called *associate classes*. Represent class R_i by an incidence matrix A_i. The R_is comprise an *association scheme on* X if

1. $\sum_{i=0}^{m} A_i = J$
2. for all i, $A_i^\top = A_i$
3. for all i, j, $\exists\, p_{ij}^k \in \mathbb{N}$ such that $A_i A_j = \sum_k p_{ij}^k A_k$.

A *partially balanced incomplete block design* $\mathrm{PBIBD}(m)$ with parameters $v, b, r, k, \lambda_1, \ldots, \lambda_m$ based on X has b blocks, all of size k, each $x \in X$ occurs in exactly r blocks, and if $(x, y) \in R_i$ then x, y occur together in exactly λ_i blocks.

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Lemma

Let N be an incidence matrix of a $\mathrm{PBIBD}(m)$ with parameters $v, b, r, k, \lambda_1, \ldots, \lambda_m$ arising from an association scheme with associate matrices A_i. Then

$$NN^\top = rI + \sum_{i=1}^{m} \lambda_i A_i \qquad \text{and} \qquad JN = kJ. \qquad (*)$$

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Conversely, a $v \times b$ $(0,1)$-matrix N such that $(*)$ holds for associate matrices A_i of an association scheme is an incidence matrix of a $\mathrm{PBIBD}(m)$ with parameters $v, b, r, k, \lambda_1, \ldots, \lambda_m$.

Suppose that $|G| = 4t + 2$. Using the lemma we can prove

Theorem

If $\psi \in Z^2(G, \langle -1 \rangle)$ *is quasi-orthogonal then* $\mathcal{E}_{M_\psi}$ *is an incidence matrix of a* $\mathrm{PBIBD}(4)$

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If $\psi \in Z^2(G, \langle -1 \rangle)$ is quasi-orthogonal then $\mathcal{E}_{M_\psi}$ is an incidence matrix of a $\mathrm{PBIBD}(4)$ with parameters $v = b = 8t + 4$, $r = k = 4t + 2$, $\lambda_1 = 0$, $\lambda_2 = 2t + 1$, $\lambda_3 = 2t + 2$, and $\lambda_4 = 2t$.

The established theory of quasi-orthogonal cocycles yields the converse.

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$Z^2(G, \langle -1 \rangle)$ has quasi-orthogonal elements $\iff$ there is a $\mathrm{PBIBD}(4)$ with the above parameters, on which a quasi-Hadamard group E such that $E/\langle -1 \rangle \cong G$ acts regularly, and which is R_1-class regular wrt $\langle -1 \rangle$.

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So, quasi-orthogonal cocycles always exist $\equiv$ these $\mathrm{PBIBD}(4)$s always exist.

The above equivalence also implies transposability.

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Also, if M is a $\{\pm 1\}$-matrix whose determinant attains the Ehlich-Wojtas bound, then some Hadamard equivalent of M commutes with its transpose.