# *n*-fold unbiased bases: an extension of the MUB condition

Máté Farkas

University of Gdańsk

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Jakub Borkała

Edgar Aguilar

Richard Küng

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    - in $2^d \to 1$ QRAC, the optimal measurements are MUBs

$$\{|y_i\rangle\langle y_i|\}_{i=1}^d \quad \text{and} \quad \{|z_i\rangle\langle z_i|\}_{i=1}^d$$

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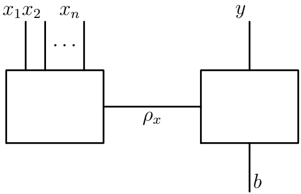


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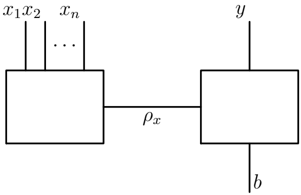


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- Figure of merit: average success probability:

$$\bar{p} = \frac{1}{nd^n} \sum_{x,y} \mathbb{P}(B = x_y \mid X = x \cap Y = y)$$

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- Example: $n = 3$, 3UB-condition:

$$\langle y_{x_y} | z_{x_z} \rangle \langle z_{x_z} | a_{x_a} \rangle \langle a_{x_a} | y_{x_y} \rangle + \langle y_{x_y} | a_{x_a} \rangle \langle a_{x_a} | z_{x_z} \rangle \langle z_{x_z} | y_{x_y} \rangle = \frac{2}{d^2} \quad \forall x_y, x_z, x_a \in [d]$$

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  - Lévy's lemma: concentration of measure on $S^{2d-1}$ $\implies$ functions admit their expectation value with high probability, when the dimension is high

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