Generalized binary sequences from quasi-orthogonal cocycles.

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Outline

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Definitions

By a **binary sequence** of length n we mean an element of $\{-1,1\}^n$. For a sequence ϕ, we denote by $\phi(k)$ the k-th entry of ϕ (starting with $k=0$).

$$\phi := (\phi(0), \phi(1), \ldots, \phi(n-1)).$$

Therefore $\phi\colon \mathbb{Z}_n \longrightarrow \{-1,1\}$ is a set map.

Definitions

For an integer w with $0 \leq w < n$, let

$$R_\phi(w) = \sum_{k=0}^{n-1} \phi(k)\phi(k+w)$$

be the **(periodic) autocorrelation** of ϕ at shift w. In the formula above $k+w$ has to be considered modulo n.

The autocorrelation sequence of ϕ is defined as

$$R_\phi = (R_\phi(0), ..., R_\phi(n-1)),$$

with $R_\phi(0)$ being **the peak-value** and all other values being **off-peak values**.

Optimal autocorrelation

It is well-known that all periodic autocorrelations of a binary sequence of length n are congruent to $n \mod 4$. Moreover,

$$\max_{0<w<n} |R_\phi(w)| \geq \begin{cases} 0 & \text{for } n \equiv 0 \mod 4 \\ 1 & \text{for } n \equiv 1 \text{ or } 3 \mod 4 \\ 2 & \text{for } n \equiv 2 \mod 4 \end{cases} \quad (1)$$

It is said that a binary sequence ϕ is **optimal** if equality holds in (1).

Optimal autocorrelation. The case $n \equiv 0 \mod 4$

If $max_{0<w<n}|R_\phi(w)| = 0$, the sequence ϕ is called **perfect**. The only known perfect binary sequence up to equivalence is $(+++-)$. It is conjectured that there is no perfect binary sequence of length greater than 4.

As an alternative to the lack of examples of perfect binary sequences:

- Perfect binary arrays (PBAs).
- Almost perfect sequences.
- ...

Definitions

Let $\mathbf{s} = (s_1, \ldots, s_r)$ be a vector of integers greater than one and let $G = \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$. **A binary s-array** of energy $n = \prod_{i=1}^{r} s_i$ is a set map $\phi\colon G \to \langle -1 \rangle$. Let us observe that if $r = 1$, ϕ is a sequence.

ϕ is called a **Perfect Binary Array (PBA(s))** if $0 \neq \mathbf{g} \in G$ implies

$$R_\phi(\mathbf{g}) = \sum_{\mathbf{j} \in G} \phi(\mathbf{j})\phi(\mathbf{g} + \mathbf{j}) = 0.$$

If ϕ is a PBA($\mathbf{s}$) then $n = \prod_{i=1}^{r} s_i = 4k^2$ for some integer k.

Generalized Perfect Binary Arrays

Let $\mathbf{z} = (z_1, \ldots, z_r)$ where $z_i = 0$ or 1. Let

$$\mathbf{G} = \mathbb{Z}_{(z_1+1)s_1} \times \cdots \times \mathbb{Z}_{(z_r+1)s_r}.$$

Further define the following subgroups of $\mathbf{G}$,

$$\begin{aligned} H &= \{\mathbf{h} \in \mathbf{G} \colon \; h_i = 0 \text{ if } z_i = 0; \;\; h_i = 0 \text{ or } s_i \text{ if } z_i = 1\} \\ K &= \{\mathbf{k} \in H \colon \; \mathbf{k} \;\text{ has even weight}\}. \end{aligned}$$

Any $\mathbf{g} \in \mathbf{G}$ may be written uniquely in the form $\mathbf{g} = \mathbf{l} + \mathbf{h}$ where $\mathbf{l} \in G$ and $\mathbf{h} \in H$.

Generalized Perfect Binary Arrays

Let $\phi\colon G \to \langle -1 \rangle$ any set function. **The expansion of ϕ with respect to z** is the function $\phi'\colon \mathbf{G} \to \langle -1 \rangle$ defined by

$$\phi'(\mathbf{g}) = \begin{cases} \phi(\mathbf{l}) & \text{if } \mathbf{h} \in K \\ -\phi(\mathbf{l}) & \text{if } \mathbf{h} \notin K, \end{cases}$$

where $\mathbf{g} = \mathbf{l} + \mathbf{h}$.

The expansion of ϕ with respect to $\mathbf{z}$

As **an example** consider $\mathbf{s} = (2,2)$, $\mathbf{z} = (1,0)$ and

$$\phi\colon \overbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}^{G} \longrightarrow \{-1,1\}$$

$$\begin{matrix} + & + \\ + & - \end{matrix},$$

then

$$\phi'\colon \overbrace{\mathbb{Z}_4 \times \mathbb{Z}_2}^{\mathbf{G}} \longrightarrow \{-1,1\}$$

$$\begin{matrix} + & + \\ + & - \\ - & - \\ - & + \end{matrix}$$

where $H = \{(0,0),(2,0)\}$ and $K = \{(0,0)\}$

Definitions

If ϕ is a binary sequence, that is, $\phi\colon \mathbb{Z}_n \longrightarrow \{-1,1\}$, then **the expansion of** ϕ (with respect to $\mathbf{z}=1$) is the concatenation of ϕ and $-\phi$.

$$\phi' := (\phi(0),\ldots,\phi(n-1),-\phi(0),\ldots,-\phi(n-1)).$$

Definitions

$\phi \colon G \longrightarrow \{-1,1\}$ is called a **Generalized Perfect binary array**, GPBA($\mathbf{s}$), of type $\mathbf{z}$, if $\mathbf{g} \in \mathbf{G} - H$ implies

$$R_{\phi'}(\mathbf{g}) = \sum_{\mathbf{j} \in \mathbf{G}} \phi'(\mathbf{j})\phi'(\mathbf{g}+\mathbf{j}) = 0.$$

Let $\mathbf{s} = (2,2)$, $\mathbf{z} = (1,0)$ and $\phi \colon \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \{-1,1\}$

$$\begin{matrix} + & + \\ + & - \end{matrix},$$

Then ϕ is a GPBA(2,2) of type $\mathbf{z} = (1,0)$.

Known results. Jedwab, Hughes, Horadam ...

ϕ	$n = \prod s_i$	*Cocycles*	Difference sets
PBA($\mathbf{s}$)	$4k^2$	$\partial\phi$ is an orthogonal coboundary over $G = \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$	Menon-Hadamard DS
GPBA($\mathbf{s}$) of type $\mathbf{z}$	2 or $4k$	$f_J\partial\phi$is an orthogonal cocycle over $G = \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$	$(n, 2, n, n/2)$-RDS in G/K relative to H/K

Question. The case $n \equiv 2 \mod 4$

ϕ	$n = \prod s_i$	$R_\phi(w)$	*Cocycles*	Difference sets
Optimal binary sequences (and arrays)	$4k+2$	± 2	?	Almost DS Arasu, Ding. . . (2001)
GOBA($\mathbf{s}$) of type $\mathbf{z}$???	?	?	?	?

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Negaperiodic Golay pairs

Balonin, Dokovic. Negaperiodic Golay pairs and Hadamard matrices. In. Control Syst. 5, 2–17 (2015)

The **negaperiodic autocorrelation function** of a sequence ϕ and shift w is

$$\mathrm{NR}_\phi(w) = \phi \cdot \phi N^w$$

where N is the the negacyclic matrix given by

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Negaperiodic Golay pairs

Balonin, Dokovic. Negaperiodic Golay pairs and Hadamard matrices. In. Control Syst. 5, 2–17 (2015)

A pair (ϕ_1, ϕ_2) of sequences of length $2t$ is called a **negaperiodic Golay pair (NGP)** if

$$\mathrm{NR}_{\phi_1}(w) + \mathrm{NR}_{\phi_2}(w) = 0, \quad \text{for all}\, 1 \leq w \leq 2t-1.$$

FACTS:

- NGPs are **the associated pairs** in Ito's terminology (2000).
- A NGP of length $2t$ can be used to directly **construct a Hadamard matrix** of order $4t$.
- Let ϕ a sequence of length $2t$. Then

$$R_{\phi'}(w) = 2\mathrm{NR}_{\phi}(w), \quad \text{for all}\, 0 \leq w \leq 2t-1.$$

Negaperiodic Golay pairs

Property

A pair (ϕ_1, ϕ_2) of sequences of length $2t$ is a **negaperiodic Golay pair (NGP)** if and only if

$$R_{\phi'_1}(w) + R_{\phi'_2}(w) = 0, \quad \text{for all } 1 \leq w \leq 2t-1.$$

The pair (ϕ'_1, ϕ'_2) is called an **extended NGP** by Egan in

R. Egan. On equivalence of negaperiodic Golay pairs. Des. Codes Cryptogr. 1–10 (2016)

Negaperiodic Golay pairs

FACTS:

- By definition, if ϕ is a Generalized Perfect Binary Sequence (GPBS) of length $2t$ then $R_{\phi'}(w) = 0$, for all $1 \leq w \leq 2t-1$.
- As a consequence, every pair (ϕ_1, ϕ_2) of GPBS of length $2t$ is a NGP.

EXAMPLE: $\phi = (+,+)$ is a GPBS since $\phi' = (+,+,-,-)$ and $R_{\phi'} = (4,0,-4,0)$. Then (ϕ,ϕ) is a NGP.

$$\Phi = \begin{bmatrix} + & + \\ - & + \end{bmatrix}, \quad H = \begin{bmatrix} \Phi_1 & \Phi_2 \\ -\Phi_2^T & \Phi_1^T \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ - & + & - & + \\ - & + & + & - \\ - & - & + & + \end{bmatrix}.$$

Negaperiodic Golay pairs

A NEGATIVE RESULT:

Result 4.8 of J. Jedwab. Generalized perfect arrays and Menon Hadamard difference sets. Des. Codes Cryptogr. 2, 19–68 (1992)

ϕ is a GPBS of length n iff $n = 2$.

QUESTION: What kind of sequence could be a good alternative to GPBS?

- If ϕ is a sequence of length $2t$ (and $t > 1$), then

$$\max_{1 \leq w \leq 2t-1} |R_{\phi'}(w)| \geq 4.$$

- Therefore, we choose as a "good" alternative to GPBS a sequence ϕ such that

$$\max_{\mathbf{1 \leq w \leq 2t-1}} |\mathbf{R}_{\phi'}(\mathbf{w})| = \mathbf{4}.$$

Negaperiodic Golay pairs

We say that ϕ is a **Generalized Optimal Binary Sequence** of length $2t$ if for all $1 \leq w \leq 2t-1$,

t odd

$$|R_{\phi'}(w)| = \begin{cases} 0 & \text{if } w \text{ is odd} \\ 4 & \text{if } w \text{ is even} \end{cases}$$

t even

$$|R_{\phi'}(w)| = \begin{cases} 4 & \text{if } w \text{ is odd} \\ 0 & \text{if } w \text{ is even} \end{cases}$$

Here, we will deal with the case **t odd**.

Negaperiodic Golay pairs

Example: Let $\mathbf{t} = \mathbf{3}$, $\phi_1 = (+,-,+,+,+,+)$ and $\phi_2 = (+,+,-,+,+,+)$. Then

$$\begin{array}{l} \phi_1' = (+,-,+,+,+,+,-,+,-,-,-,-), \\ \phi_2' = (+,+,-,+,+,+,-,-,+,-,-,-); \end{array}$$

and

$$\begin{array}{l} R_{\phi_1'} = (12,\mathbf{0},\quad \mathbf{4},\mathbf{0},-\mathbf{4},\mathbf{0},-12,0,-4,0,\quad 4,0), \\ R_{\phi_2'} = (12,\mathbf{0},-\mathbf{4},\mathbf{0},\quad \mathbf{4},\mathbf{0},-12,0,\quad 4,0,-4,0). \end{array}$$

Therefore, ϕ_1 and ϕ_2 are **GOBS** of length 6. Moreover, (ϕ_1, ϕ_2) is a **NGP** since $R_{\phi_1'}(w) + R_{\phi_2'}(w) = 0$, for all $1 \leq w \leq 5$.

Negaperiodic Golay pairs

Examples

t	ϕ_1	ϕ_2
3	$(1^2, 4)$	$(2, 1, 3)$
5	$(2, 1^3, 5)$	$(3, 1, 2, 1, 3)$
7	$(2, 1, 5, 1^3, 3)$	$(2, 1, 4, 2, 1^2, 3)$

Table: NGPs from GOBS of length $2t$, $t = 3, 5, 7$.

Notation $$(\overset{1}{+}, \overset{1}{-}, \overbrace{+, +, +, +}^{4}) = (1, 1, 4) = (1^2, 4).$$

$2t \equiv 2 \mod 4$ – Analogous to GPBAs

Questions

- How to find GOBSs?... and NGPs from GOBSs?
- How to extend the definition of GOBS to arrays (GOBAs)?

An approach:

- Try to look for equivalent combinatorial objects: Cocycles, Difference sets ...

Question. The case $n \equiv 2 \mod 4$

ϕ	$n = \prod s_i$	$R_\phi(w)$	*Cocycles*	Difference sets
Optimal binary sequences (and arrays)	$4k+2$	± 2	?	Almost DS Arasu, Ding... (2001)
GOBA(**s**) of type **z**???	?	?	?	?

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Cocycles and Cocyclic matrices

Let $G = \{g_1, g_2, \dots, g_n\}$ be a group of order n.

- $f: G \times G \to C = \{-1, 1\}$ such that

$$f(g_i g_j, g_k) f(g_i, g_j) = f(g_j, g_k) f(g_i, g_j g_k), \quad \forall g_i, g_j, g_k \in G.$$

- $M_f = [f(g_i, g_j)]$ binary matrix coming from f indexed by $G \times G$.

Example

$$\begin{array}{rccl} f: & \mathbb{Z}_2 \times \mathbb{Z}_2 & \rightarrow & C = \{-1,1\} \\ & (a,b) & \mapsto & f(a,b) = (-1)^{ab} \quad \text{is a cocycle} \end{array}$$

$$M_f = \begin{pmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

This is the Sylvester-Hadamard matrix of order 2.

Computing Cocyclic matrices

Given a group $G = \{g_1, \ldots, g_n\}$ and a base of cocycles over G.

$$B = \text{Representatives} \bigcup \text{Coboundaries}$$

Then,

$$M_f = R \cdot M_{\delta_{i_1}} \cdots M_{\delta_{i_w}}$$

where R is a product of representative cocycles and $M_{\delta_{i_j}}$ are coboundaries.

Cocylic approach: Searching for Hadamard matrices

Cocyclic Hadamard test (1992, de Launey-Horadam)

	General	Cocyclic
Time	$O(t^3)$	$O(t^2)$
Space	$O(2^{n^2})$	$O(2^n)$

The “Cocyclic Hadamard” conjecture (1993, de Launey-Horadam)

- There exist Cocyclic Hadamard Matrices of order $4t$ for all t.

- The smallest order for which no cocyclic Hadamard matrix is known is 188.

RE of M

Definition

Let $M = (m_{i,j})$ be a cocyclic matrix of order n.

$$\mathrm{RE}(M) = \sum_{i>1} |\sum_{j} m_{i,j}|$$

Proposition

Let $M = (m_{i,j})$ be a cocyclic matrix of order n.

- $\mathrm{RE}(M) = 0$ iff M is a cocyclic Hadamard matrix. The associated cocycle is called **orthogonal**.
- If $n \equiv 2 \mod 4$ then $\mathrm{RE}(M) \geq 2 \cdot (\frac{n}{2} - 1)$.

RE of M

Proposition

Let $M = (m_{i,j})$ be a cocyclic matrix of order n.

- $\mathrm{RE}(M) = 0$ iff M is a cocyclic Hadamard matrix. The associated cocycle is called **orthogonal**.
- If $n \equiv 2 \mod 4$ then $\mathrm{RE}(M) \geq 2 \cdot (\frac{n}{2} - 1)$.

Definition

Let G be a group of order n with $n \equiv 2 \mod 4$. Then $\psi \in Z^2(G, \langle -1 \rangle)$ is **quasi-orthogonal** if $\mathrm{RE}(M_\psi) = 2 \cdot (\frac{n}{2} - 1)$.

GOBSs and quasi-orthogonal cocycles

Theorem

There exists an explicit constructive equivalence between Generalized Optimal Binary Sequences and quasi-orthogonal cocycles over $\mathbb{Z}_{2t}$ with t odd.

GOBSs and quasi-orthogonal cocycles

Theorem

There exists an explicit constructive equivalence between Generalized Optimal Binary Sequences and quasi-orthogonal cocycles over $\mathbb{Z}_{2t}$ with t odd.

Proof of existence of quasi-orthogonal cocycles over all G of orders $2k+2$ $\forall k \geq 2$ may be within reach. Existence verified computationally for all G of orders ≤ 42 (so far; E. O'Brien).

Example

(R. Egan.) Take any Hadamard matrix with circulant core and let A be the normalized core. Then $\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right] \otimes A$ displays a quasi-orthogonal cocycle.

Example, $n = 6$, GOBSs from quasi-orthogonal cocycles

1. Compute a quasi-orthogonal cocycle,

$$M_\psi = \begin{bmatrix} + & + & + & + & + & + \\ + & + & - & - & - & + \\ + & - & + & + & - & + \\ + & - & + & - & + & - \\ + & - & - & + & - & - \\ + & + & + & - & - & - \end{bmatrix}$$

2. Write M_ψ in terms of the elements of a base.

$$M_\psi = \begin{bmatrix} + & + & + & + & + & + \\ + & + & + & + & + & - \\ + & + & + & + & - & - \\ + & + & + & - & - & - \\ + & + & - & - & - & - \\ + & - & - & - & - & - \end{bmatrix} \begin{bmatrix} + & + & + & + & + & + \\ + & + & - & - & - & - \\ + & - & + & + & + & - \\ + & - & + & + & - & + \\ + & - & + & - & + & + \\ + & - & - & + & + & + \end{bmatrix}$$

3. We obtain the GOBS. $\phi = (+, -, +, +, +, +)$

Negaperiodic Golay pairs from quasi-orthogonal cocycles

t	ϕ_1	ϕ_2
3	$(1^2, 4)$	$(2, 1, 3)$
5	$(2, 1^3, 5)$	$(3, 1, 2, 1, 3)$
7	$(2, 1, 5, 1^3, 3)$	$(2, 1, 4, 2, 1^2, 3)$
9	$(3, 1, 2, 1^3, 3, 1, 5)$	$(2, 1, 2, 3, 2, 1^3, 5)$
11	$(3, 2, 2, 1, 2, 1^5, 7)$	$--$
13	$(3, 2, 2, 1, 2, 1^5, 7)$	$(3, 3, 1, 3, 1, 2, 1, 2, 1^4, 2)$

Table: NGPs from GOBS of length $2t$.

Generalized Optimal Binary Arrays

Definition. We call ϕ a *GOBA*($\mathbf{s}$) *of type* $\mathbf{z}$ if $\mathbf{g} \in \mathbf{G} - H$ implies

$$R_{\phi'}(\mathbf{g}) = \sum_{\mathbf{j} \in \mathbf{G}} \phi'(\mathbf{j})\phi'(\mathbf{g}+\mathbf{j}) \in \{-2|H|, 0, 2|H|\},$$

such that

$$\text{If } z_1 = 0, \quad |R_{\phi'}^{-1}(0)| = 0, \quad |R_{\phi'}^{-1}(-2|H|)| = |R_{\phi'}^{-1}(2|H|)| = \frac{|\mathbf{G}|-|H|}{2}$$
$$\text{If } z_1 = 1, \quad |R_{\phi'}^{-1}(0)| = \frac{|\mathbf{G}|}{2}, \quad |R_{\phi'}^{-1}(-2|H|)| = |R_{\phi'}^{-1}(2|H|)| = \frac{\frac{|\mathbf{G}|}{2}-|H|}{2},$$

where $R_{\phi'}^{-1}(n) = \{\mathbf{g} \in \mathbf{G} - H\colon R_{\phi'}(\mathbf{g}) = n\}$.
If $\mathbf{z} = \mathbf{0}$ the above definition reduces to: $\mathbf{0} \neq \mathbf{l} \in G$ implies

$$\sum_{\mathbf{j} \in G} \phi(\mathbf{j})\phi(\mathbf{l}+\mathbf{j}) = \pm 2.$$

GOBAs and quasi-orthogonal cocycles

Theorem

There exists an explicit constructive equivalence between Generalized Optimal Binary Arrays and quasi-orthogonal cocycles over an abelian group of order $2t$ with t odd.

Conclusions

ϕ	$n = \prod s_i$	$R_\phi(w)$	*Cocycles*	Difference sets
Optimal binary sequences (and arrays)	$4k+2$	± 2	quasi-orthogonal coboundaries over finite abelian groups	Almost Difference sets Arasu, Ding. . . (2001)
GOBA(**s**) of type **z**	$4k+2$	$-2\|H\|, 0, 2\|H\|$	quasi-orthogonal cocycles over finite abelian groups	relative quasi-difference sets

Quasi-orthogonal coboundaries

A **coboundary** is a cocycle of the form $\partial\phi$ where $\partial\phi(g,h) = \phi(g)\phi(h)\phi(gh)$ for some (normalized) map $\phi\colon G \to \langle -1 \rangle$.

Lemma

If $|G| = 4t + 2$ and $\psi \in Z^2(G, \langle -1 \rangle)$ is a coboundary

$$RE(M_\psi) \geq 8t + 2. \quad (2)$$

We say that a coboundary ψ is **quasi-orthogonal**, if equality holds in (2).

Quasi-orthogonal coboundaries

Lemma

If $|G| = 4t + 2$ *and* $\psi \in Z^2(G, \langle -1 \rangle)$ *is a coboundary, if and only if,* $|\{g \in G \setminus \{1\} : \sum_{h \in G} \psi(g,h) = \pm 2\}| = 4t + 1$.

Example

$$G = \mathbb{Z}_6, \qquad M_\psi = \begin{bmatrix} + & + & + & + & + & + \\ + & - & + & - & - & - \\ + & + & + & - & - & + \\ + & - & - & + & - & - \\ + & - & - & - & - & + \\ + & - & + & - & + & + \end{bmatrix}$$

Quasi-orthogonal coboundaries

Theorem

Suppose G abelian and $|G| = 4t + 2$. Let D be a subset of G of cardinality k, with characteristic function $f: G \to GF(2)$ and $\phi(x) = (-1)^{f(x)}$. Define $R^* = \{(\phi(g), g) : g \in G\} \subset \mathbb{Z}_2 \times G$. Then the following statements are equivalent.

i. *The coboundary $\partial\phi$ is quasi-orthogonal.*

ii. *D is a $(4t + 2, k, k - (t + 1), (4t + 2)k - k^2 - (4t + 1)t)$-ADS in G.*

iii. *R^* is a normal "extremal" relative $(4t + 2, 2, 4t + 2, 2t + 1)$-quasi difference set in $\mathbb{Z}_2 \times G$ relative to $\mathbb{Z}_2 \times 1$.*

iv. *$f: G \to GF(2)$ has optimum nonlinearity.*

v. *$\phi: G \to \{\pm 1\}$ is a binary array with optimal autocorrelation. If G is the cyclic group, then ϕ is a binary sequence with*

Thank you!!!

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