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# Ricci solitons on compact three-manifolds

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*Abstract*: In this short article we show that there are no compact three-dimensional Ricci solitons other than spaces of constant curvature. This generalizes a result obtained for surfaces by Hamilton [4]. The proof involves a careful analysis of the ODE for the curvature which is associated to the Ricci flow.

*Keywords*: Ricci flow.

*MS classification*: 53C20, 53C21.

## Introduction

In this paper we are concerned with the Ricci flow for Riemannian metrics

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g)$$

and, when the underlying manifold $M^n$ is compact, the normalized Ricci flow

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) + 2\frac{r}{n}g, \tag{1}$$

where $r$ is the integral of the scalar curvature divided by the volume. (The normalization fixes the volume of $M$.) The Ricci flow, introduced by Richard Hamilton [2], has proved useful in extending classification results in Riemannian and Kähler geometry. In general, one starts with a metric $g_0$ on $M$ that satisfies some rather general curvature condition $\mathcal{C}$, and proves that as the normalized Ricci flow runs, the metrics $g_t$ converge to a limiting metric which satisfies a more restrictive condition $\mathcal{C}'$ (see, for example, [3, 5,6]). In fact, the most well-known results assume the curvature of $g_0$ is positive (in some sense) and show convergence to constant curvature.

The case of compact surfaces is different. Since, in dimension two, the Ricci flow only changes the metric by a conformal factor, and one might expect it to converge to

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the constant curvature metric appropriate to the topology of the surface. Indeed, this was demonstrated in [4] and [1]; there, the bulk of the analysis is devoted to the case $M = S^2$, and this could be summarized as

- prove long-time existence of the flow
- prove convergence to a soliton
- classify solitons on $S^2$.

A *soliton* for the normalized Ricci flow (1) is a metric that changes only by pullback by a one-parameter family of diffeomorphisms as it evolves under (1). It turns out that this is equivalent to the initial metric satisfying

$$\mathcal{L}_X g = -2\,\mathrm{Ric}(g) + 2\frac{r}{n} g$$

for some vector field $X$. Classifying solitons on $S^2$ is not too hard once one sees that $X$ must be conformal and a divergence. The only vector field available forces a soliton metric to have an $S^1$ symmetry, and by classifying solutions to an ODE, it follows that the only soliton on $S^2$ has constant curvature (see [4, §10]).

In this paper we extend the classification of solitons to compact three-manifolds. We show that, in fact, the only solitons are constant curvature metrics.

## Solitons and breathers

In general, a "breather" is a solution to an evolution equation that is periodic over time. For our purposes, it means a solution to the normalized Ricci flow that is periodic, up to diffeomorphism; that is, $g_T = \phi^* g_0$ for some fixed period $T$ and diffeomorphism $\phi$. It follows that breathers, like solitons, have uniformly bounded curvature and volume. It will turn out that in dimension three the Ricci flow does not admit any non-trivial breathers either.

**Proposition 1.** *Any soliton or breather on a compact connected manifold is either Einstein with nonpositive Ricci curvature, or has positive scalar curvature.*

**Proof.** The scalar curvature obeys

$$\frac{\partial R}{\partial t} = \Delta R + 2|\,\mathrm{Rt}\,|^2 + \frac{2}{n} R(R - r) \tag{2}$$

where 'Rt' is the traceless part of the Ricci tensor. Consider a point in space and time where $R$ is at a global minimum. If $R < 0$ there, then $\Delta R \geqslant 0$ and $\partial R/\partial t = 0$ force $\mathrm{Rt} = 0$ and $R = r$ there; hence $R$ is constant over $M$ at this time. By applying the same argument at each point of $M$ at this time, we the metric is Einstein. Otherwise, $R \geqslant 0$ always.

Assume $R$ has minimum value zero; using (2), we know that $\Delta R \leqslant 0$ at this point. Let $\Omega$ be the open set where $\Delta R < 0$ at this time. If $\Omega$ is not empty, then $R$ can only achieve its infimum—zero—on the boundary of $\Omega$, and by the Hopf maximum principle the outward normal derivative of $R$ is negative there. This contradicts $dR = 0$ there.

Then $\Omega$ is empty, and since $R$ achieves its maximum somewhere, $R$ is constant—in fact, identically zero—by the maximum principle, and the metric is Einstein by the same argument as above.

It follows that solitons and breathers have a positive lower bound for the scalar curvature. Now we can apply the main result of this paper:

**Theorem 2.** *If $g_t$ obeys the normalized Ricci flow on a compact $M^3$ and has scalar curvature bounded below by a positive constant $R_0$ for all $t \geqslant 0$, then there exists a function $\phi(t) > 0$ such that* $\mathrm{Ric}(g_t) \geqslant -\phi(t)\, g_t$ *and* $\lim_{t\to\infty} \phi(t) = 0$.

The proof of this theorem will depend on applying a maximum principle to the curvature tensor. We will actually run the unnormalized flow on the same initial metric, since the corresponding evolution equations for the curvature are simpler and we know how to go back and forth between the two flows. We will show that as the manifold shrinks to a point, the minimum as well as the maximum of the scalar curvature approach infinity. This is important since we will construct a pinching set—that is, a set defined by inequalities on the curvature that are preserved by the flow—that forces the lowest eigenvalue of Ric at a point to go towards zero as $R \to \infty$ at that point.

**Theorem 3.** *If $g_t$ obeys the normalized Ricci flow on compact $M^3$ and has* $\mathrm{Ric} \geqslant 0$, *and average scalar curvature $r$ bounded above for all $t \geqslant 0$, then $g_t$ converges to a metric of constant positive curvature.*

**Proof.** In [3], Hamilton has shown that either the Ricci curvature becomes positive immediately, or $M$ splits locally as a product of a one-dimensional flat factor and a surface with positive curvature, and this splitting is preserved by the flow. In the former case, we know $g_t$ converges to constant positive curvature. To rule out the latter case, consider the evolution equation for $r$ under the normalized flow:

$$\frac{d}{dt}\int R\mu_g = -\int \Big\langle \frac{\partial g}{\partial t}, \mathrm{Ric} - \frac{1}{2}Rg \Big\rangle \mu_g.$$

(Here we have fixed the volume to be one.) We can calculate the integrand on the right pointwise by diagonalizing the Ricci tensor. We obtain $\langle 2rg/n - 2\,\mathrm{Ric}, \mathrm{Ric} - Rg/2\rangle \leqslant -rR/3$ and $dr/dt \geqslant r^2/3$. This would mean $r$ increases without bound, contradicting our assumption.

**Corollary 4.** *There are no three-dimensional solitons or breathers on a compact connected $M^3$ other than constant curvature metrics.*

**Proof.** The metrics $g_t$ must have $r$ bounded, and by the proposition above, $R > 0$ or else the metric is Einstein, i.e. constant curvature. If $R > 0$ then, by our main result, the lower bound for the lowest eigenvalue of the Ricci tensor goes to zero from below as $t \to \infty$. But since the minimum over $M$ of the lowest eigenvalue is a periodic function of time, it must have been at least zero to begin with. Now Theorem 3 completes the proof.

In the remainder of this paper we will prove Theorem 2.

## Controlling the scalar curvature

Suppose $s$ is the time coordinate under the normalized flow, and let $g_t$ on $M^n$ be the unnormalized flow with the same initial data. Then $g_s$ and $g_t$ differ by a change in scale and a reparametrization in time: if $\psi(t) = \text{vol}(g_t)^{-2/n}$ then $g_s$ is given by

$$s = \int_0^t \psi(t)dt, \quad g_s = \psi(t)g_t.$$

We assume $g_s$ evolves with scalar curvature bounded above $R_0 > 0$. Consider the evolution equation for the volume of $g_t$:

$$\frac{d}{dt}\log \text{vol}(g_t) = -r(g_t) = -r(g_s)\psi(t) \leqslant -R_0 \,\text{vol}(g_t)^{-2/n}.$$

Since the integrals of the ordinary differential equation $d/dt\ \log v = -R_0 v^{-2/n}$ go to zero in finite time, we know that $\text{vol}(g_t)$ hits zero at some finite time $T$. (Note that $g_t$ remains smooth until time $T$ since $g_s$ is smooth.) Since $R(g_t) \geqslant \psi(t)R_0$, the scalar curvature of $g_t$ goes to infinity everywhere on $M$ at time $T$.

## Some three-dimensional geometry

On a three-dimensional oriented inner product space $V$, the volume form gives us an isometry between $V$ and $\Lambda^2 V$. Under this isometry, an orthonormal basis $(e_1, e_2, e_3)$ is carried to $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2)$. Given any quadratic form on $\Lambda^2 V$, we can find an orthonormal basis of $V$ so that the form is diagonalized in the corresponding basis for $\Lambda^2 V$. What this means for Riemannian geometry is that at every point $p$ of a three-dimensional Riemannian manifold $(M, g)$ we can choose an orthonormal basis for $T_pM$ that diagonalizes the curvature tensor; that is,

$$R_{2323} = m_1, \quad R_{3131} = m_2, \quad R_{1212} = m_3$$

and all other components are zero.

The $m_i$ are the eigenvalues of the curvature operator. It is easy to check that in this frame the Ricci tensor is also diagonalized. We will assume

$$m_1 \leqslant m_2 \leqslant m_3;$$

then $m_1 + m_2 \leqslant m_1 + m_3 \leqslant m_2 + m_3$ are the eigenvalues of the Ricci tensor, and $R = 2(m_1 + m_2 + m_3)$.

## The maximum principle

Rather than evolving a metric $g$ on the tangent bundle, we will use the "Uhlenbeck trick" (see [3]), that is, evolve a gauge transformation on $TM$ so that $g$ pulls back to be a fixed metric $h$. If we diagonalize the curvature tensor Rm with respect to $g$ at a point, we also diagonalize the its pullback $\widetilde{\mathrm{Rm}}$ with respect to $h$. Since the metric on $V = \Lambda^2 T^* M$ is fixed, the following maximum principle applies to $\widetilde{\mathrm{Rm}}$.

**Theorem 5** (Hamilton [3]). *Let $V$ be a vector bundle on a compact manifold $M$ and $h$ be a fixed metric on $V$. Suppose $g$ is a metric on $M$ and $\nabla$ a connection on $V$ compatible with $h$, both possibly varying in time. Let $\phi$ be a vector field on $V$ tangent to the fibers. Assume $X$ is a closed subset of $V$, convex in each fibre, invariant under parallel translation at all times, and such that solutions of the ODE $ds/dt = \phi(s)$ for sections of $V$ remain inside $X$. Then solutions of the heat equation $\partial s/\partial t = \Delta s + \phi(s)$ also remain inside $X$.*

Fortunately, when we diagonalize the curvature, the right-hand side of the ODE that corresponds to the Uhlenbeck-normalized evolution equation for curvature is also diagonalized. Now we can write down a system of ODE for the eigenvalues of the curvature:

$$\dot{m}_1 = m_1^2 + m_2 m_3, \quad \dot{m}_2 = m_2^2 + m_1 m_3, \quad \dot{m}_3 = m_3^2 + m_1 m_2.$$

Since the right-hand side is homogeneous in the $m_i$, dilation in space and time is a symmetry of this system. This means that the integral curves of this system in $\mathbb{R}^3$ project onto a well-defined set of trajectories on the unit sphere (see Figure 1).

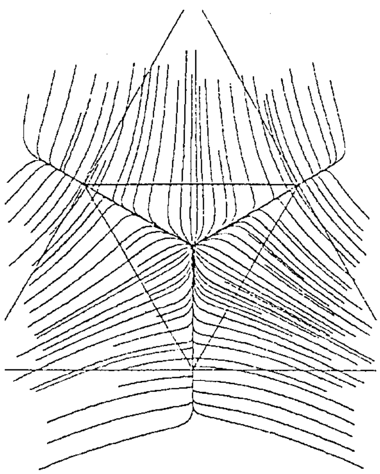

Fig. 1. The projection is

$$x = \frac{m_1 - m_2}{\sqrt{3}(m_1 + m_2 + m_3)}, \quad y = \frac{m_1 + m_2}{m_1 + m_2 + m_3} - \frac{2}{3}.$$

The large triangle is Ric $\geqslant 0$, and the small triangle is Rm $\geqslant 0$.

We will just be interested in the behaviour of the ODE restricted to the wedge defined $m_1 \leqslant m_2 \leqslant m_3$ and $m_1 + m_2 + m_3 > 0$. There, the integral curves are attracted either to the line $m_1 = m_2 = m_3$ or to the line $m_1 = m_2 = 0$.

## The pinching set

Our set $X$ will be given by $m_1 + m_2 \geqslant -2f(z)$ for a positive function $f$ of $z = m_1 + m_2 + m_3 = R/2$. Since $X$ is defined in terms of eigenvalues relative to $h$ it will be invariant under parallel translation. We will require

1. $f''(z) \leqslant 0$, in order for $X$ to be convex in each fibre;
2. $X$ to be preserved by the ODE; and
3. $\lim_{z\to\infty} f(z)/z = 0$.

This will then force $(m_1 + m_2)/R$, which is a dilation-invariant quantity, to be at least zero as $R \to \infty$.

To compute condition (2), note that

$$\frac{d}{dt}(m_1 + m_2 + 2f(z)) \\ = (m_1 + m_2)z - 2m_1m_2 + 2(z^2 - (m_1 + m_2)m_3 - m_1m_2)f'.$$

Along the boundary of $X$ given by $m_1 + m_2 = -2f(z)$ we need this derivative to be nonnegative. Assume $f' > 0$; then for a fixed $z$ the derivative is minimized when $m_2 = m_1$. Then our condition for $f$ becomes

$$f' \geqslant \frac{m_1^2 - m_1 z}{z^2 - 2m_1 z + 3m_1^2} = \frac{(f+z)f}{2f^2 + (z+f)^2}.$$

By making $df/dz$ larger than necessary we get a simpler ODE for $f$. Choose

$$f' = \frac{f}{f+z}.$$

Solutions of this ODE are invariant under the dilation $f \to \lambda f$, $z \to \lambda z$; and, once $f$ is positive it is an increasing function of $z$. Thus we can choose $f$ so that $\inf_M m_1 + m_2 \geqslant -f(\inf_M z)$ for the initial metric $g_0$, so that the curvature of $g_0$ lies inside $X$.

To see that $f'' \leqslant 0$, let $p = z/f$. Then

$$\frac{dp}{dz} = \frac{1}{f+z} = \frac{p}{p+zp}.$$

Since $p$ is monotone increasing, $f' = 1/(p+1)$ is decreasing. Finally, by separation of variables $\log p + p = \log z + C$, so $\lim_{z\to\infty} p = \infty$; thus $f/z \to 0$.

This ends the proof of Theorem 2.

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